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Multivariate Differences, Polynomials, and Splines

THOMAS KUNKLE*

Department of Mathematics, College of Charleston, Charleston, South Carolina
E-mail: kunkle@math.cofc.edu

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We generalize the univariate divided difference to a multivariate setting by considering linear combinations of point evaluations that annihilate the null space of certain differential operators. The relationship between such a linear functional and polynomial interpolation resembles that between the divided difference and Lagrange interpolation. Applying the functional to the shifted multivariate truncated power produces a compactly supported spline by which the functional can be represented as an integral. Examples include, but are not limited to, the tensor product B-spline and the box spline. © 1996 Academic Press, Inc.

1. INTRODUCTION

Denote the divided difference of a univariate function f at the points $x_0 < x_1 < \cdots < x_n$ by $[x_0, ..., x_n] f$, and the nth derivative of f by $D^n f$. We state without proof the following familiar properties of the divided difference.

(1.1) *If* $D^n f \equiv 0$, *then* $[x_0, ..., x_n] f = 0$.

More specifically, if f is continuous on the interval $[x_0, x_n]$ and if $D^n f$ is identically zero on (x_0, x_n), then the divided difference of f is zero.

(1.2) *There exist scalars* $\lambda(i)$ *depending on* $\{x_0, ..., x_n\}$ *such that* $[x_0, ..., x_n] f = \sum \lambda(i) f(x_i)$.

Thus, $[x_0, ..., x_n]$ is a said to be a linear combination of point evaluations on $\{x_0, ..., x_n\}$.

(1.3) $[x_0, ..., x_n] f = 0$ *iff there exists a polynomial of degree less than n agreeing with f on* $x_0, ..., x_n$.

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(1.4) *$M(t \mid x_0, ..., x_n) := n[x_0, ..., x_n](\cdot - t)_+^{n-1}$ is a nonnegative piecewise polynomial that is compactly supported and $n-2$ times continuously differentiable.*

One commonly refers to M as a **B-spline**.

(1.5) *If $D^n f$ is continuous on $[x_0, x_n]$, then $n!\,[x_0, ..., x_n]\, f = \int_{\mathbb{R}} D^n f(t)\, M(t \mid x_0, ..., x_n)\, dt$.*

(1.6) *If $D^n f$ is continuous on the interval $[x_0, x_n]$, then $n!\,[x_0, ..., x_n]\, f = D^n f(\xi)$ for some $\xi \in (x_0, x_n)$.*

(1.7) *$n!\,[x_0, ..., x_n]\, f \to D^n f(\xi)$ as $x_0, ..., x_n \to \xi$. This convergence is accelerated if the divided differences are centered about ξ.*

The divided differences and associated B-splines have been generalized in many ways to a multivariate setting.

The simplest generalization to d variables is made via tensor products. For $1 \leqslant i \leqslant d$, let $\alpha(i)$ be a nonnegative integer, and let $x_{0,i}, x_{1,i}, ..., x_{\alpha(i),i}$ be distinct points in $\mathbb{R}$. Define $\bigotimes [x_{0,i}, x_{1,i}, ..., x_{\alpha(i),i}]\, f$ to be the result of applying $[x_{0,i}, x_{1,i}, ..., x_{\alpha(i),i}]$ to the ith variable of f for every i. One can prove tensor product analogues of (1.1)–(1.7). For instance, in place of (1.2), we have that the tensor product divided difference of f depends linearly on f's values on the rectangular grid of points

$$S = \{x \colon \forall i,\ x(i) \in \{x_{0,i}, x_{1,i}, ..., x_{\alpha(i),i}\}\}.$$

The nth derivative D^n is replaced throughout by the mixed partial

$$D^\alpha := \prod_{i=1}^{d} \left(\frac{\partial}{\partial x(i)}\right)^{\alpha(i)},$$

and the B-spline is replaced by the tensor product B-spline

$$M(t \mid S) := \prod_{i=1}^{d} M(t(i) \mid x_{0,i}, x_{1,i}, ..., x_{\alpha(i),i}).$$

Comparing the tensor product with the univariate case, in which $\{x_0, ..., x_n\}$ are arbitrary, one sees its the primary limitation: the requirement that the knot sequence S be a rectangular grid of points. Other generalizations of the divided difference and B-spline have done away with this requirement.

Introduced by de Boor [1] with an attribution to Schoenberg, the multivariate B-spline $M(t \mid S)$ has been studied extensively by, among others, Dahmen, Michelli, and Höllig [4, 6–10, 13, 14]. Dahmen [6] introduces the multivariate divided difference and multivariate truncated power and proves identities similar to (1.1), (1.2), and (1.4)–(1.7). He replaces D^n by a product of directional derivatives and makes no restrictions on S. In contrast to (1.2), however, the multivariate divided difference of f depends linearly on not only f but also its derivatives. Though the multivariate B-spline and divided difference satisfy an identity similar to (1.4), the spline is a linear combination of translates of more than one truncated power. This can be avoided, it seems, only if one places further restrictions on S.

Both Kergin [15] and Hakopian [11, 12] have studied multivariate polynomial interpolation and, in doing so, generalized the divided difference. Kergin applies a homogeneous differential operator to a smooth function and integrates the result against the multivariate B-spline [6]. He places no restriction on S. Hakopian [11] defines a d-variate αth divided difference to be the integral of $D^\alpha f$ against $(\alpha!)^{-1} M(\cdot \mid S)$, where $|\alpha| = \# S - d$. Analogous to (1.2), if S is in general position, then the αth divided difference of a function can be expressed as a linear combination of its integrals over lower-dimensional simplices. A similar result holds for the Kergin functionals in case the homogeneous differential operator is a product of directional derivatives [17]. In the interpolation problem using these integrals as conditions, the generalized divided differences play a role that reduces to (1.3) in the univariate case.

More recently, Neamtu [18, 19, 20] defines a divided difference of a smooth function on $\mathbb{R}^{d \times d}$, and proves a (1.4)-like relation between his functional, the multivariate B-spline, and a generalized truncated power function (different from Dahmen's). Neamtu proves certain recurrence relations for his divided difference and therefore for the multivariate B-spline.

Along another track, the box spline, introduced by de Boor and DeVore [2], generalizes the univariate cardinal B-spline to several variables; results analogous to (1.1)–(1.7) hold when one replaces D^n by the product of directional derivatives [5]. The linear difference functional depends solely on the values of f on the finite point set S, which is required to lie on a not-necessarily-rectangular grid with uniform spacing in each of its (d or more) directions.

This raises the question as to whether one can obtain results like (1.1)–(1.7) with a linear functional made up of point evaluations on S and still allow S to be more general than in the box spline or tensor product settings. To answer this, we replace D^n by an arbitrary product D_{N} of directional derivatives and define (Definition 3.1) an "Nth difference" to be a functional satisfying the natural generalizations of (1.1) and (1.2). Corollary 3.3 gives a necessary geometric condition on the support of any Nth difference. The relationship between Nth differences and polynomial interpolation generalizes (1.3); see Theorem 3.5 and Corollary 3.7. As in (1.4) and (1.5), applying an Nth difference to the multivariate truncated power produces a compactly supported spline by which the functional can be represented as an integral. Details are in Theorem 3.15 and its corollary. Lemma 4.13 generalizes (1.6) and Lemma 4.14 and Theorem 4.15 generalize (1.7).

In addition, we prove other properties and characterizations of Nth differences and the accompanying splines and polynomial spaces. Examples include, but are not limited to, the tensor product B-spline and the box spline.

We begin by establishing some notation in the next section.

2. Notation

The symbol S is reserved to stand for a set of finitely many points in $\mathbb{R}^d$. It is convenient to think of $\mathbb{R}^d$ as consisting of column vectors. Thus, letting $s^\top$ denote the transpose of $s \in \mathbb{R}^d$, the inner product of s and r in $\mathbb{R}^d$ is $s^\top r$. For H a subset of $\mathbb{R}^d$, we let $H^\perp$ denote the space of all r in $\mathbb{R}^d$ perpendicular to everything in H.

Let δ_s denote the **point-evaluation** at s, that is, the functional given by $\delta_s f = f(s)$. For $\lambda = \sum_S \lambda(s)\,\delta_s$, the support of λ, denoted $\operatorname{supp}\lambda$, is the set of $s \in S$ for which $\lambda(s) \neq 0$. Denote the convex hull of S by $[\![S]\!]_1$, and denote by $[\![S]\!]_+$ the cone $\{\sum x(s)s\colon \forall s \in S, x(s) \geqslant 0\}$.

The letter N shall always stand for a matrix whose typical column is (the nonzero vector) $v \in \mathbb{R}^d$. We shall borrow the following notation from the box spline literature [5]: since the order of its columns is unimportant for our purposes, one can think of N simply as a multiset in $\mathbb{R}^d \backslash \{0\}$. This eliminates the need for indices other than the elements of N itself. For instance, interpretting A^B in the standard way, $\mathbb{R}^{\mathrm{N}}$ denotes the set of all functions from N into $\mathbb{R}$ (i.e., the set of all real vectors indexed by N), and the map N is defined by

$$\mathrm{N}\colon \mathbb{R}^{\mathrm{N}} \to \mathbb{R}\colon x \mapsto \mathrm{N}x := \sum_v v x(v).$$

The space of all linear combinations of elements or columns of N, that is, the image of the above map, is denoted $\operatorname{ran}\mathrm{N}$. The cone defined earlier, $[\![\mathrm{N}]\!]_+$, is just the image of $[0, \infty)^{\mathrm{N}}$ under this same map. Let $\alpha(v)$ denote the multiplicity of v in N, i.e., the number of columns in N that are identical to v;

The directional derivative in the direction v is denoted D_v. Define

$$D_{\mathrm{N}} := \prod D_v.$$

This derivative is said to exist if it exists unambiguously, independent of the order of differentiation. For $\mathscr{U}$ a closed set in $\mathbb{R}^d$, let $C^{\mathrm{N}}(\mathscr{U})$ denote the set of functions f for which $D_{\mathrm{K}} f$ is continuous on $\mathscr{U}$ for every $\mathrm{K} \subseteq \mathrm{N}$; i.e., $D_{\mathrm{K}} f$ exists on the interior $\mathscr{U}^0$ of $\mathscr{U}$ and is uniquely extendible by continuity to all finite boundary points. The space $C_c^{\mathrm{N}}(\mathbb{R}^d)$ shall consist of those $f \in C^{\mathrm{N}}(\mathbb{R}^d)$ having compact support.

3. Nth Differences

We shall consider linear functionals which satisfy (1.1) and (1.2) in a general sense.

Definition 3.1. Let λ be a linear combination of point evaluations with $\operatorname{supp} \lambda = S$, a finite set in $\mathbb{R}^d$, and let N be a matrix whose columns lie in $\mathbb{R}^d \backslash \{0\}$. We say that λ is an Nth **difference** if there exists a closed disk $\mathscr{U}$ containing S such that $\lambda f = 0$ for every f in $C^{\mathrm{N}}(\mathscr{U})$ whose Nth derivative $D_{\mathrm{N}} f$ is identically zero on $\mathscr{U}^0$.

For $v \in \mathrm{N}$, define the relations $x \equiv^v y$ and $x \equiv^{\mathrm{N}} y$ for $x, y \in \mathbb{R}^d$ to mean that $x - y$ lies in $\operatorname{ran} v$ or $\operatorname{ran} \mathrm{N}$, respectively. Let $(s : v)$ and $(s : \mathrm{N})$ denote the sets of $t \in S$ such $t \equiv^v s$ or $t \equiv^{\mathrm{N}} s$, respectively. Define the restrictions

$$\lambda|_{(s:v)} : f \mapsto \sum_{(s:v)} \lambda(t) f(t)$$

and

$$\lambda|_{(s:\mathrm{N})} : f \mapsto \sum_{(s:\mathrm{N})} \lambda(t) f(t).$$

Let $v^{\alpha(v)}$ denote the multiset of $\alpha(v)$ copies of v.

For λ to be an Nth difference implies that the restrictions above are differences in their own right.

Theorem 3.2. *Let λ be an* N*th difference with support S. Then for every v in* N *and s in S,*

(a) $\lambda|_{(s:v)}$ *is a $v^{\alpha(v)}$th difference, and*

(b) $\lambda|_{(s:\mathrm{N})}$ *is an* N*th difference.*

If the columns of N span $\mathbb{R}^d$, we shall say that N is **complete**. Part (b) implies that, if N is not complete, then λ is a sum of Nth differences with supports on lower-dimensional hyperplanes. Consequently, any statement about complete Nth differences and their relationship to $\mathbb{R}^d$ has as an immediate corollary a corresponding statement about incomplete Nth differences and their relationship to these hyperplanes.

Proof of Theorem 3.2. Definition 3.1 requires that, if we wish to prove (a) or (b), we first choose disks containing $(s:v)$ and $(s:\mathrm{N})$. For these, take the disk $\mathscr{U}$ containing S associated to λ by that definition.

To prove (a), assume that f is in $C^{v^{\alpha(v)}}(\mathscr{U})$ and that $D_v^{\alpha(v)} f=0$ on $\mathscr{U}^0$. Then there exists a univariate polynomial p so that $D^{\alpha(v)}p=0$ and $f(t)=p(v^\top t)$ for all $t\in(s:v)$.

Let Q be the orthogonal projector from $\mathbb{R}^d$ onto $v^\perp$; then $Qx=Qy$ if and only if $x\equiv^v y$. Consequently, there exists $g\in C^\infty(\mathbb{R}^d)$ satisfying

$$(g\circ Q)(t)=\begin{cases}1 & \text{if}\quad t\in(s:v),\\ 0 & \text{if}\quad t\in S\backslash(s:v).\end{cases}$$

The directional derivative $D_v(g\circ Q)$ is identically zero on $\mathbb{R}^d$, as one can check directly.

The product $F:=(p\circ v^\top)(g\circ Q)$ is zero on $S\backslash(s:v)$, agrees with f on $(s:v)$, and belongs to $C^{\mathrm{N}}(\mathscr{U})$. (In fact, F is infinitely differentiable on all of $\mathbb{R}^d$.) Furthermore, its Nth derivative is identically zero, since

$$D_v^{\alpha(v)}F=(D^{\alpha(v)}p\circ v^\top)(g\circ Q).$$

Therefore $0=\lambda F=\lambda|_{(s:v)}f$, proving (a).

To prove (b), suppose that $f\in C^{\mathrm{N}}(\mathscr{U})$ satisfies $D_{\mathrm{N}}f=0$ on $\mathscr{U}^0$.

If rank $\mathrm{N}=d$, the proof is trivial, since $(s:\mathrm{N})=S$. Assume rank $\mathrm{N}<d$. Let Q be the orthogonal projector from $\mathbb{R}^d$ onto $\mathrm{N}^\perp$, and let $g\in C^\infty(\mathbb{R}^d)$ satisfy

$$(g\circ Q)(t)=\begin{cases}1 & \text{if}\quad t\in(s:\mathrm{N}),\\ 0 & \text{if}\quad t\in S\backslash(s:\mathrm{N}).\end{cases}$$

Then $D_v(g\circ Q)$ is identically zero for any v in N.

The product $F:=f(g\circ Q)$ agrees with f on $(s:\mathrm{N})$ and is zero on $S\backslash(s:\mathrm{N})$, and its Nth derivative is identically zero on $\mathscr{U}^0$. Hence $0=\lambda F=\lambda|_{(s:\mathrm{N})}f$, completing the proof of (b). ∎

For which sets S do there exist Nth differences supported within S? Theorem 3.2 implies the following necessary condition on S.

COROLLARY 3.3. *If λ is a nontrivial* N*th difference, and if* supp $\lambda\subset S$, *then for every s in S and v in* N, *the equivalence class $(s:v)$ must have more than $\alpha(v)$ members.*

The converse is proven false in the next section.

Proof. It will suffice to prove this in case supp $\lambda = S$, so that $\lambda(s) = 0$ for no s in S.

Let λ be an Nth difference with support S, and suppose that, for some s and v, the set $(s:v)$ has $\alpha(v)$ or fewer members. Then there exists a univariate polynomial p of degree less than $\alpha(v)$ such that

$$p(v^\top t) = \begin{cases} 0 & \text{if} \quad t \in (s:v)\backslash s, \\ 1 & \text{if} \quad t = s. \end{cases}$$

Therefore $\lambda|_{(s:v)} p \circ v^\top = \lambda(s)$. On the other hand, since $D_v^{\alpha(v)}(p \circ v^\top)$ is identically zero, $\lambda|_{(s:v)} p \circ v^\top = 0$, a contradiction. ▮

Extending the above in a natural way to complex d-space, we can choose

$$N = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{3.4}$$

so that D_{N} is the Laplacian. Corollary 3.3 then implies that there is no nontrivial Nth difference supported in $\mathbb{R}^2$, since $s \in \mathbb{R}^2$ implies that $(s:v)$ has only one member for v equal either $\binom{1}{i}$ or $\binom{1}{-i}$. There are, of course, many useful discretizations of the Laplacian, typically a sum of Nth differences for

$$D_{\mathrm{N}} = \left(\frac{\partial}{\partial x(1)}\right)^2 \qquad \text{and} \qquad D_{\mathrm{N}} = \left(\frac{\partial}{\partial x(2)}\right)^2.$$

Let Π denote the space of polynomials of d variables, and define Π_{N} to be the space of all polynomials p for which $D_{\mathrm{N}} p = 0$.

For example, if N is as in (3.4), then $\Pi_{\mathrm{N}}(\mathbb{R}^2)$ consists of all harmonic polynomials of two real variables. The impossibility of a nontrivial Laplacian Nth difference is also a consequence of the next theorem, since one can always fit a harmonic polynomial to prescribed values on a finite set in the plane.

THEOREM 3.5. *Let λ be linear combination of finitely many point evaluations on S. Then λ is an* N*th difference if and only if $\lambda p = 0$ for all $p \in \Pi_{\mathrm{N}}$.*

Proof. Clearly, if λ is an Nth difference, then $\lambda \perp \Pi_{\mathrm{N}}$. To prove the converse, suppose $\lambda \perp \Pi_{\mathrm{N}}$.

Let $\mathcal{U}$ be any closed disk containing S and let $f \in C^{\mathrm{N}}(\mathcal{U})$ satisfy $D_{\mathrm{N}} f = 0$. Since $\lambda f = \lambda(f - p)$ for every $p \in \Pi_{\mathrm{N}}$, in order to show that $\lambda f = 0$ it will suffice to prove that, for every positive ε, there is a $p \in \Pi_{\mathrm{N}}$ such that $\|f-p\| < \varepsilon$, where $\|\cdot\|$ is the sup-norm on $\mathcal{U}$.

The proof is by induction on the number of elements in N and uses the density of Π among continuous functions on any compact set [16].

Suppose first that $\mathrm{N} = \{v\}$. Let u be the center of $\mathcal{U}$ and let

$$A\colon x \mapsto u + \left(1 - \frac{vv^{\top}}{v^{\top}v}\right)(x-u),$$

the affine map that projects $\mathcal{U}$ onto the the $(d-1)$-dimensional disk $\mathcal{U}_v := (u + v^{\perp}) \cap \mathcal{U}$. If f is in $C^{v}(\mathcal{U})$ and $D_v f = 0$, then $f = f \circ A$ on $\mathcal{U}$. Take p a polynomial such that $|f(x) - p(x)| < \varepsilon$ for all $x \in \mathcal{U}_v$. Then $p \circ A \in \Pi_v$ and $\|f - p \circ A\| < \varepsilon$, proving Theorem 3.5 when $\mathrm{N} = \{v\}$.

For the inductive step, choose $v \in \mathrm{N}$, and assume that f in $C^{\mathrm{N}}(\mathcal{U})$ satisfies $D_{\mathrm{N}} f = 0$. Applying the inductive hypotheses to $D_v f$ gives, for every $\varepsilon > 0$, a polynomial q in $\Pi_{\mathrm{N}\backslash v}$ such that $\|D_v f - q\| < \varepsilon$. For all $x \in \mathcal{U}$,

$$f(x) = f(A(x)) + \int_{A(x)}^{x} D_v f,$$

where $\int$ denotes a line integral. Choose $p_1 \in \Pi$ such that $\|f \circ A - p_1 \circ A\| < \varepsilon$. Define

$$p_2(x) := \int_{A(x)}^{x} q.$$

Then $p_2 \in \Pi$, and $D_v p_2 = q$. Set $p = p_1 \circ A + p_2$. Then $p \in \Pi_{\mathrm{N}}$, and

$$\|f - p\| \leqslant \|f \circ A - p_1 \circ A\| + \left\| p_2 - \int_{A(\cdot)}^{\cdot} D_v f \right\| < (1 + \text{radius } \mathcal{U})\, \varepsilon.$$

Since the above is true for all $\varepsilon > 0$, the proof is complete. ∎

Since $\mathrm{K} \subset \mathrm{N}$ implies $\Pi_{\mathrm{K}} \subset \Pi_{\mathrm{N}}$, Theorem 3.5 has the following immediate corollary.

Corollary 3.6. *If λ is an* N*th difference, and if* $\mathrm{K} \subset \mathrm{N}$, *then λ is an* K*th difference.*

Define $\Pi_{\mathrm{N}}(S) := \{p|_S \colon p \in \Pi_{\mathrm{N}}\}$. The next corollary is a generalization of (1.3).

Corollary 3.7. *Let f be defined on the finite set S. Then $f \in \Pi_{\mathrm{N}}(S)$ if and only if $\lambda f = 0$ for every* N*th difference λ with* $\operatorname{supp} \lambda \subset S$.

In other words, $\lambda f = 0$ for all Nth differences on S if and only if there exists a polynomial in Π_{N} agreeing with f on S.

Proof. Since $\Pi_{\mathrm{N}}(S)$ is a closed subspace of $\mathbb{R}^S$, it can be written as

$$\bigcap \left\{ \ker \lambda : \lambda = \sum_{S} \lambda(s)\, \delta_s,\ \lambda \perp \Pi_{\mathrm{N}}(S) \right\}.$$

By Theorem 3.5, this is

$$\bigcap \{ \ker \lambda : \lambda \text{ an Nth difference, } \operatorname{supp} \lambda \subset S \},$$

completing the proof. ∎

We say that N is a **directional matrix** and write $\mathrm{N} \in \mathbb{D}^{d \times m}$ if it contains no distinct parallel elements and if N satisfies either of the equivalent conditions below.

(3.8) *The convex hull* $[\![\mathrm{N}]\!]_1$ *does not contain the origin.*

(3.9) *The elements of* N *lie in some open half-plane; i.e.,* $\exists \gamma \in \mathbb{R}^d$ *such that* $\gamma^{\top} v > 0$ *for all* $v \in \mathrm{N}$.

For example, both of

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfy (3.8), but only the first belongs to $\mathbb{D}^{2 \times 3}$.

For σ in $\{-1, 1\}^{\mathrm{N}}$, we set $\mathrm{N}_\sigma := \{\sigma(v) v : v \in N\}$. If N has no distinct parallel elements, and $0 \notin \mathrm{N}$, then there exists at least one σ for which $\mathrm{N}_\sigma \in \mathbb{D}^{d \times m}$. Such a σ is called a **normalization** of N. Clearly, λ is an Nth difference if and only if it is a N_σth difference for every normalization of N.

Next, we briefly review properties of the multivariate truncated power T_{N} [6]. For N a directional matrix the Nth **truncated power**, T_{N}, is the distribution whose inner product with any test function ϕ is defined by

$$\langle T_{\mathrm{N}}, \phi \rangle := \int_{[0, \infty)^{\mathrm{N}}} \phi(\mathrm{N} t)\, dt.$$

For completeness, define $T_{\varnothing}$ to be the Dirac δ-distribution. Either condition (3.8) or (3.9) guarantees the existence of this integral.

Clearly T_N has support $[\![N]\!]_+$. Furthermore, T_N is a homogeneous piecewise polynomial distribution of total degree $m-d$. For example, when the members of N are linearly independent, T_N is the piecewise constant distribution

$$T_N(x) = \begin{cases} |\det N|^{-1} & \text{if} \quad x \in [\![N]\!]_+; \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

The smoothness of T_N depends on N's "degeneracy": N is s-**degenerate** if each of its subsets of cardinality $d+s$ has rank d. If N is complete and s-degenerate, then T_N is exactly $m-s-d-1$ times continuously differentiable.

If $K \subset N$, then

$$T_K * T_{N\backslash K} = T_N \qquad \text{and} \qquad D_K T_N = T_{N\backslash K}. \tag{3.11}$$

In particular, $D_N T_N = \delta$, so that $D_N(T_N * \phi) = \phi$ for any test function ϕ. In fact, for the convolution $T_N * f$ to serve as a Nth antiderivative of f, it is sufficient that f be a continuous function of compact support, as one can easily prove. We include this fact in the following lemma.

LEMMA 3.12. *Let* $N \in \mathbb{D}^{d \times m}$, *and let* f *be a continuous function of compact support in* $\mathbb{R}^d$. *Then* $T_N * f \in C^N(\mathbb{R}^d)$, *and* $D_N(T_N * f) = f$ *on* $\mathbb{R}^d$.

(Note that, in the above, $T_N * f$ is continuous even in directions other than those in N.)

We next combine an Nth difference and the truncated power to form a piecewise polynomial (distribution) in a way that generalizes (1.4).

DEFINITION 3.13. Let N be a directional matrix. For λ an Nth difference with support S, define the (Nth) **representer** of λ to be the distribution

$$M(t \mid \lambda, N) := \lambda T_N(\cdot - t) = \sum_S \lambda(s)\, T_N(s-t).$$

If N is complete, $M(\cdot \mid \lambda, N)$ is piecewise polynomial function, having the same total degree and smoothness as T_N. Otherwise, M is supported on hyperplanes spanned by N passing through S. As a consequence of Theorem 3.2, part b (see the remarks thereafter), one need only consider complete directional matrices in order to thoroughly understand the relationship between any Nth difference and its representer. For instance, compare Corollary 3.16 and Corollary 3.17.

LEMMA 3.14. *Let* N *be a complete directional matrix. Let* λ *be an* N*th difference and* $\mathscr{U}$ *the associated closed disk containing the support* S *of* λ. *Then, for every* f *in* $C^{\mathrm{N}}(\mathscr{U})$,

$$\lambda f = \int_{\mathscr{U}} M(\cdot \mid \lambda, \mathrm{N})\, D_{\mathrm{N}} f.$$

Proof. Given ε positive, let $\mathscr{U}^\varepsilon$ be the ball concentric with $\mathscr{U}$ having radius ε greater than the radius of $\mathscr{U}$. Let F_ε be a continuous function on $\mathbb{R}^d$ that agrees with $D_{\mathrm{N}} f$ on $\mathscr{U}$, is supported on $\mathscr{U}^\varepsilon$, and satisfies

$$\max_{\mathscr{U}^\varepsilon} |F_\varepsilon| \leqslant \max_{\mathscr{U}} |D_{\mathrm{N}} f|.$$

By Lemma 3.12, $T_{\mathrm{N}} * F_\varepsilon \in C^{\mathrm{N}}(\mathscr{U})$, and its Nth derivative agrees with $D_{\mathrm{N}} f$ on $\mathscr{U}^0$. Therefore, Definition 3.1 implies that

$$\lambda f = \lambda(T_{\mathrm{N}} * F_\varepsilon) = \int_{\mathbb{R}^d} M(\cdot \mid \lambda, \mathrm{N})\, F_\varepsilon.$$

Letting $\varepsilon \to 0$ and applying the Dominated Convergence Theorem yields the desired result. ∎

Like the B-spline, $M(\cdot \mid \lambda, \mathrm{N})$ is compactly supported. The next theorem covers this in detail.

THEOREM 3.15. *Let* N *be a complete directional matrix. Let* λ *be an* N*th difference with support* S.

(a) *If* σ *is any normalization of* N, *then* $M(t \mid \lambda, \mathrm{N}) = M(t \mid \lambda, \mathrm{N}_\sigma) \prod_{\mathrm{N}} \sigma(v)$, *at all points* t *at which both functions are continuous.* (*Since* N *is complete, this is almost everywhere.*)

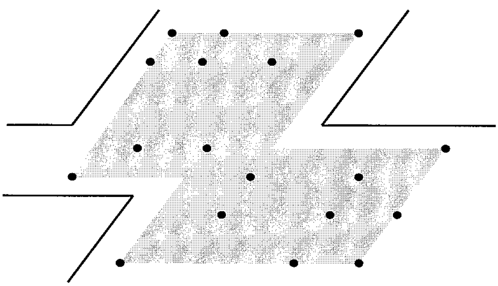

FIGURE 1

(b) *If* $(x+[\![\mathrm{N}_\sigma]\!]_+)\cap S=\varnothing$ *for some normalization* σ *of* N, *then* $M(x \mid \lambda, \mathrm{N})=0$.

(c) *The support of* $M(\cdot \mid \lambda, \mathrm{N})$ *lies within the convex hull* $[\![S]\!]_1$.

With the vector $(1,0)^\top$ corresponding to the horizontal, and $(0,1)^\top$ to the vertical, Fig. 1 illustrates (b) and (c) in case S is the set of points marked by a • and

$$\mathrm{N}=\begin{pmatrix}1 & 1 & 2 & 2\\ 0 & 0 & 3 & 3\end{pmatrix}.$$

For any s in S and v in N, the class $(s:v)$ consists of all points in S collinear with s in the direction v. Each of the directions v in N appears with multiplicity two, so according to Corollary 3.3, each $(s:v)$ must contain at least three elements. If λ is an Nth difference supported on S, then part (b) of Theorem 3.15 implies that the support of $M(\cdot \mid \lambda, \mathrm{N})$ is contained in the shaded region, since at any other point one can place a cone $[\![\mathrm{N}_\sigma]\!]_+(\angle)$ not intersecting S. This region is a subset of the convex hull $[\![S]\!]_1$, as promised by part (c).

Proof of Theorem 3.15. Since both N and N_σ are directional matrices, and since λ is both an Nth and an N_σth difference, Lemma 3.14 implies

$$\int_{\mathscr{U}} M(\cdot \mid \lambda, \mathrm{N})\, D_{\mathrm{N}} f=\int_{\mathscr{U}} M(\cdot \mid \lambda, \mathrm{N}_\sigma)\, D_{\mathrm{N}_\sigma} f$$

$$=\int_{\mathscr{U}} M(\cdot \mid \lambda, \mathrm{N}_\sigma) \prod_{\mathrm{N}} \sigma(v)\, D_{\mathrm{N}} f$$

for every f in $C^{\mathrm{N}}(\mathscr{U})$. Lemma 3.12 guarantees that, by choosing f in $C^{\mathrm{N}}(\mathscr{U})$ wisely, $D_{\mathrm{N}} f$ can be made equal to any nonnegative hat function with compact support in $\mathscr{U}$. Therefore the functions $M(\cdot \mid \lambda, \mathrm{N})$ and $M(\cdot \mid \lambda, \mathrm{N}_\sigma) \prod \sigma(v)$ must agree at any points within $\mathscr{U}$ at which both functions are continuous.

If $\mathscr{W}$ is any closed disk containing $\mathscr{U}$, then $f\in C^{\mathrm{N}}(\mathscr{W})$ and $D_{\mathrm{N}} f=0$ on $\mathscr{W}$ implies $\lambda f=0$, so we can replace $\mathscr{U}$ by $\mathscr{W}$ is all our conclusions. Thus $M(\cdot \mid \lambda, \mathrm{N})$ agrees with $M(\cdot \mid \lambda, \mathrm{N}_\sigma) \prod \sigma(v)$ at all points of continuity within any closed disk $\mathscr{W}\supset\mathscr{U}$, and therefore these functions are identical anywhere they both are continuous in $\mathbb{R}^d$. (The parenthetical remark in part (a) refers to the fact that T_{N} is piecewise continuous when N spans $\mathbb{R}^d$.) This completes the proof of (a).

Since the support of T_{N_σ} is $[\![\mathrm{N}_\sigma]\!]_+$, the support of $M(\cdot \mid \lambda, \mathrm{N}_\sigma)$ lies within $S-[\![\mathrm{N}_\sigma]\!]_+$. This set does not contain x by hypothesis, so $M(x \mid \lambda, \mathrm{N})=0$, proving (b).

To see that supp $M \subset [\![S]\!]_1$, let x lie outside $[\![S]\!]_1$, and consider the set of all $\gamma \in \mathbb{R}^d$ satisfying $\gamma^\top x > \gamma^\top s$ for all $s \in S$. By the Separation Corollary, this set is nonempty, and since it is open, it has positive measure. Those γ for which $\gamma^\top v = 0$ for some $v \in \mathrm{N}$ form a set of measure zero, so there exists γ satisfying $\gamma^\top x > \gamma^\top s$ for every s and $\gamma^\top v \neq 0$ for every v.

Let $\sigma(v) := \operatorname{sign}(\gamma^\top v)$. Then $\gamma^\top \cdot$ is positive on N_σ, so σ is a normalization of N. Furthermore,

$$\min(\gamma^\top(x + [\![\mathrm{N}_\sigma]\!]_+)) = \gamma^\top x > \max(\gamma^\top S),$$

implying that $(x + [\![\mathrm{N}_\sigma]\!]_+) \cap S$ is empty. By part (b), $M(x \mid \lambda, \mathrm{N}) = 0$, proving (c). ∎

One consequence of Theorem 3.15 is that, since the convex hull $[\![S]\!]_1$ lies within $\mathscr{U}$, the domain of integration in Lemma 3.14 completely contains the support of M. We summarize this in the following corollary (interpreting the integral below as one over either the support of M or the domain of f).

COROLLARY 3.16. *If f is in $C^{\mathrm{N}}(\mathscr{U})$, then*

$$\lambda f = \int_{\mathbb{R}^d} M(\cdot \mid \lambda, \mathrm{N})\, D_{\mathrm{N}} f.$$

We can extend these results to the case that N is not complete by using part (b) of Theorem 3.2.

COROLLARY 3.17. *Let* N *be a (not necessarily complete) directional matrix, and let λ be an* N*th difference with support S. If f is in $C^{\mathrm{N}}(\mathscr{U})$, then*

$$\lambda f = \langle M(\cdot \mid \lambda, \mathrm{N}), D_{\mathrm{N}} f \rangle$$

The inner product above is a sum of integrals over hyperplanes passing through S parallel to ran N.

It is not necessary that M be nonnegative. For example, if $d = 1$ and $D_{\mathrm{N}} = (d/dx)$, then $\lambda := \delta_0 - 2\delta_1 + \delta_3$ is an Nth difference, and its representer is the piecewise constant function

$$M(t \mid \lambda, N) = \begin{cases} -1 & \text{if } 0 < t < 1; \\ 1 & \text{if } 1 < t < 3. \end{cases}$$

4. Properties of Nth Differences and Their Representers

When exactly d distinct directions appear in the complete directional matrix N, an Nth difference is little more than a tensor product divided difference. The next theorem states this specifically.

Theorem 4.1. *Let* N *be a complete directional* $d \times m$ *matrix containing exactly* d *distinct elements. Then, after a linear change of variables, any* N*th difference can be written as a linear combination of tensor product* N*th divided differences.*

Proof. We will first prove this in case the distinct vectors of N form the standard orthonormal basis for $\mathbb{R}^d$. Denote by $\alpha(j)$ the multiplicity of the jth vector in this basis. Then $D_{\mathrm{N}} = D^{\alpha}$.

Let λ be an Nth difference with support contained in a finite set S. Add points to S as necessary so that it has the form

$$S = \{s\colon \forall j,\, s(j) \in \{s_{0,j}, s_{1,j}, \ldots, s_{\beta(j),j}\}\}.$$

For every multiindex $\gamma \leqslant \beta$, let s_γ denote the point

$$s_\gamma := (s_{\gamma(1),1}, s_{\gamma(2),2}, \ldots, s_{\gamma(d),d})^\top$$

in S. For $\gamma \geqslant \alpha$, let δ_γ be the tensor product divided difference

$$\delta_\gamma := \bigotimes [s_{\gamma(j)-\alpha(j),j}, \ldots, s_{\gamma(j)-1,j}, s_{\gamma(j),j}].$$

Then δ_γ is an Nth difference.

Subtract $c_\beta \delta_\beta$ from λ, where the scalar c_β is chosen so that $\lambda - c_\beta \delta_\beta$ is supported in $S \backslash s_\beta$. From this functional subtract, for every γ satisfying $\alpha \leqslant \gamma \leqslant \beta$ and $\gamma \neq \beta$, a scalar multiple of δ_γ so that the resulting Nth difference

$$\lambda - \sum_\gamma c_\gamma \delta_\gamma \tag{4.2}$$

is supported entirely in

$$T := S \backslash \{s_\gamma\colon \alpha \leqslant \gamma \leqslant \beta\}.$$

This set "contains its shadow," meaning that if $s_\gamma \in T$ and if τ is a multiindex less than or equal to γ, then $s_\tau \in T$. It is known that, for any function f, there is a polynomial in Π_{N} agreeing with f on T. (In fact, one can always interpolate uniquely from the space

$$\operatorname{ran}\{(\cdot)^\tau : s_\tau \in T\},$$

where $(\cdot)^\tau$ is the τth monomial [12].) Consequently, $\Pi_{\mathrm{N}}(T) = \mathbb{R}^T$. By Theorem 3.5, the functional (4.2) is identically zero, proving the theorem in this special case.

More generally, let K be the set of distinct elements in N. Then K is invertible, and the functional

$$\tilde{\lambda}: f \mapsto \lambda(f \circ \mathrm{K}^{-1})$$

is a $(\mathrm{K}^{-1}\mathrm{N})$th difference. By the special case already proven, $\tilde{\lambda} = \sum c_\gamma \delta_\gamma$, where each δ_γ is a tensor product divided difference. But then $\lambda = \sum c_\gamma \tilde{\delta}_\gamma$, where

$$\tilde{\delta}_\gamma: f \mapsto \delta_\gamma(f \circ \mathrm{K}),$$

finishing the proof. ∎

Using the above proof, we'll now see that the converse of Corollary 3.3 is false.

Let

$$S := \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,3),$$
$$(2,0), (2,2), (2,3), (4,1), (4,2), (4,3)\}$$

and

$$N := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then $\#(s:v) > \alpha(v)$ for every s and v. Suppose that λ is an Nth difference supported on S. According to the proof of Theorem 4.1, there exist constants $c_{i,j}$ so that

$$\lambda = c_{0,0}[0,1,2] \otimes [0,1,2]$$
$$+ c_{1,0}[1,2,4] \otimes [0,1,2]$$
$$+ c_{0,1}[0,1,2] \otimes [1,2,3]$$
$$+ c_{1,1}[1,2,4] \otimes [1,2,3].$$

Of the four tensor product divided differences above, only the second has support at the point $(4,0)$. Since the coefficient $\lambda(4,0) = 0$, this forces $c_{1,0} = 0$. Similarly, $c_{0,1}$ must also be zero, since $\lambda(0,3) = 0$. The first of these four has the same (nonzero) coefficient at $(1,2)$ as it does at $(2,1)$. However, the fourth has two different coefficients at $(1,2)$ and $(2,1)$.

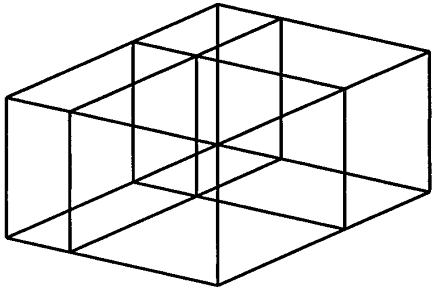

FIGURE 2

Hence the conditions $\lambda(1,2)=\lambda(2,1)=0$ lead to an invertible homogeneous system in $c_{0,0}$ and $c_{1,1}$, forcing $c_{0,0}=c_{1,1}=0$. Thus there are no (nontrivial) Nth differences supported on S.

When the complete directional matrix N contains more than d distinct elements, an analogue of Theorem 4.1 is unknown, although one can generalize the tensor product B-spline and divided difference as follows.

Let $\mathrm{N}\in\mathbb{D}^{d\times m}$ be complete, and let K be the set of distinct elements of N. Let M^α_{K} be a tensor product B-spline defined on $\mathbb{R}^{\mathrm{K}}$, having degree $\alpha(\kappa)-1$ in the κth variable, and let δ^α be the associated tensor product divided difference. Then

$$\int_{\mathbb{R}^{\mathrm{K}}} M^\alpha_{\mathrm{K}}\, D^\alpha f=\delta^\alpha f$$

for f in $C^\alpha(\mathbb{R}^{\mathrm{K}})$. Define

$$\lambda\colon C^{\mathrm{N}}(\mathbb{R}^d)\to\mathbb{R}\colon f\mapsto\int_{\mathbb{R}^{\mathrm{K}}} M^\alpha_{\mathrm{K}}(D_{\mathrm{N}}f)\circ \mathrm{K}.$$

Clearly, $\lambda p=0$ if $p\in\Pi_{\mathrm{N}}$; since

$$\lambda f=\int_{\mathbb{R}^{\mathrm{K}}} M^\alpha_{\mathrm{K}}\, D^\alpha(f\circ\mathrm{K})=\delta^\alpha(f\circ\mathrm{K}),$$

λ is a linear combination of finitely many point evaluations. Theorem 3.5 therefore guarantees that λ is an Nth difference.

Figure 2 shows the two-dimensional support and grid lines of such a bivariate spline; Figure 3 is its three-dimensional plot.

One can view the identity

$$\int_{\mathbb{R}^{\mathrm{K}}} M^\alpha_{\mathrm{K}}(x)(D_{\mathrm{N}}f)(\mathrm{K}x)\,dx=\int_{\mathbb{R}^d} M(t\mid\lambda,\mathrm{N})\,D_{\mathrm{N}}f(t)\,dt \tag{4.3}$$

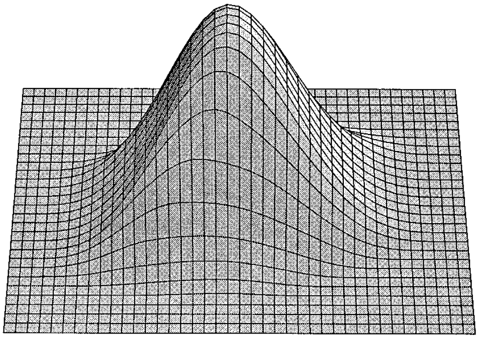

FIGURE 3

as saying that $M(\cdot \mid \lambda, \mathrm{N})$ is defined to be a d-variate density function associated, via the linear map K, to a tensor product B-spline M_{K}^{α} of $\#\mathrm{K} \geqslant d$ variables. (This has much in common with other instances throughout spline theory in which a function of several variables is used to define a function of fewer variables via an onto map such as K.) If K is the $d \times d$ identity matrix, then λ and $M(\cdot \mid \lambda, \mathrm{N})$ are simply a tensor product divided difference and B-spline. If, for every κ in K, the knots of M_{K}^{α} in the κth variable are equidistant, then the representer $M(\cdot \mid \lambda, \mathrm{N})$ is a box spline. That is, in addition to its standard definition (4.5) as a density associated to the characteristic function of a cube, the $\#\mathrm{K}$-directional box spline can be seen as in (4.3) as a density associated with a tensor product cardinal B-spline of $\#\mathrm{K}$ variables.

Define the forward difference operator ∇_{ν} by $\nabla_{\nu}: f \mapsto f(\cdot + \nu) - f$, and $\nabla_{\mathrm{N}} := \prod \nabla_{\nu}$. Then

$$\nabla_{\mathrm{N}} f = \sum_{\mathbb{Z}_{+}^{\mathrm{N}}} \nabla_{\mathrm{N}}(\beta)\, f(\cdot + \mathrm{N}\beta) \tag{4.4}$$

where $\nabla_{\mathrm{N}}(0) = (-1)^{\#\mathrm{N}}$ and $\nabla_{\mathrm{N}}(\beta) = 0$ for all but finitely many multiindices β. The box spline B_{N} is defined by its action on test functions:

$$\langle B_{\mathrm{N}}, \phi \rangle := \int_{[0,1]^{\mathrm{N}}} \phi(\mathrm{N}t)\, dt. \tag{4.5}$$

A well-known property of the box spline is that, for sufficiently smooth f,

$$\int B_{\mathrm{N}} D_{\mathrm{N}} f = \nabla_{\mathrm{N}} f(0)$$

[3]. To be specific, polynomials meet the smoothness conditions, so Theorem 3.5 implies that the functional $\delta_0 \nabla_{\mathrm{N}}: f \mapsto \nabla_{\mathrm{N}} f(0)$ is an Nth difference, and since its representer is unique, $M(\cdot \mid \lambda, \mathrm{N})$ and B_{N} are identical. In particular, if $\mathrm{N} \in \mathbb{D}^{d \times m}$, then

$$B_{\mathrm{N}}(t) = \delta_0 \nabla_{\mathrm{N}} T_{\mathrm{N}}(\cdot - t) = \sum_{\mathbb{Z}_+^{\mathrm{N}}} \nabla_{\mathrm{N}}(\beta)\, T_{\mathrm{N}}(\mathrm{N}\beta - t) \tag{4.6}$$

(a known result, stated here for later reference).

A known property of the box spline [3] holds for any representer of an Nth difference.

LEMMA 4.7. *If* N *is a directional matrix, if* λ *is an* N*th difference, and if* s *is an extreme point of* supp λ, *then, in some neighborhood of* s, *the spline* $M(\cdot \mid \lambda, \mathrm{N})$ *agrees with a scalar multiple of* $T_{\mathrm{N}_\sigma}(s - \cdot)$ *for some normalization* σ *of* N.

Proof. By Theorem 3.2, part (b), it will suffice to prove this in case N is complete.

If s is an extreme point of $S := \operatorname{supp} \lambda$, then, as in the proof of Theorem 3.15, there is a γ in $\mathbb{R}^d$ such that

$$\gamma^\top s > B := \max \gamma^\top (S \backslash s)$$

and $\gamma^\top v \neq 0$ for all $v \in \mathrm{N}$. Let $\sigma(v) := \operatorname{sign}(\gamma^\top v)$, so that N_σ satisfies (3.9). By Theorem 3.15, part (a),

$$M(\cdot \mid \lambda, \mathrm{N}) = M(\cdot \mid \lambda, \mathrm{N}_\sigma) \prod_{\mathrm{N}} \sigma(v),$$

so that, up to a scalar factor, $M(\cdot \mid \lambda, \mathrm{N})$ equals

$$\lambda(s)\, T_{\mathrm{N}_\sigma}(s - \cdot) + \sum_{S \backslash s} \lambda(t)\, T_{\mathrm{N}_\sigma}(t - \cdot). \tag{4.8}$$

The set

$$V := \{x : \gamma^\top x > B\}$$

is the desired neighborhood of s: since $\gamma^\top \cdot \leqslant B$ on supp $T_{\mathrm{N}_\sigma}(t - \cdot)$ for $t \in S \backslash s$, (4.8) reduces to $\lambda(s)\, T_{\mathrm{N}_\sigma}(s - \cdot)$ on V, completing the proof. ∎

If N is a complete directional matrix, the integral of $M(\cdot \mid \lambda, \mathrm{N})$ can be computed as follows.

Take a basis $\mathrm{K} \subset \mathrm{N}$, and choose t such that the support of $M(\cdot \mid \lambda, \mathrm{N})$ lies within the cone $t + [\![\mathrm{K}]\!]_+$. Then by (3.10),

$$\int_{\mathbb{R}^d} M(\cdot \mid \lambda, \mathrm{N}) = |\det \mathrm{K}| \int_{\mathbb{R}^d} T_{\mathrm{K}}(x - t)\, M(x \mid \lambda, \mathrm{N})\, dx.$$

By Definition 3.13, this is

$$|\det \mathrm{K}| \sum \lambda(s) \int_{\mathbb{R}^d} T_{\mathrm{K}}(x-t)\, T_{\mathrm{N}}(s-x)\, dx,$$

so that, by (3.11),

$$\int_{\mathbb{R}^d} M(\cdot \mid \lambda, \mathrm{N}) = |\det \mathrm{K}|\, \lambda T_{\mathrm{N} \cup \mathrm{K}}(\cdot - t). \tag{4.9}$$

Thus, to compute the integral of M one can use the known recurrence relations for T [6]. (The derivation of (4.9) is similar to Dahmen and Michelli's calculation of a convolution [8, (3.5)] useful in finding the inner product of multivariate B-splines.)

If λ is a $(\mathrm{N} \cup \mathrm{K})$th difference as well as an Nth difference, then the right side of (4.9) is $M(t \mid \lambda, \mathrm{N} \cup \mathrm{K})$, which, for t sufficiently large, equals zero. Thus $\int_{\mathbb{R}^d} M(\cdot \mid \lambda, \mathrm{N}) = 0$. This is generalized by the next theorem.

THEOREM 4.10. *Let* K *be a matrix of linearly independent vectors containing* η. *Let* N *and* $\mathrm{N} \cup \eta$ *be complete directional matrices, and let* λ *be a* $\mathrm{N} \cup \eta$ *difference. Then, for every* $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^{\mathrm{K}}} M(x + \mathrm{K}t \mid \lambda, \mathrm{N})\, dt = 0.$$

Proof. Note that $M(\cdot \mid \lambda, \mathrm{N})$ exists since, by Corollary 3.6, λ is an Nth difference.

We begin with the claim that

$$\forall x \in \mathbb{R}^d, \qquad \int_{-\infty}^{\infty} M(x + \eta t \mid \lambda, \mathrm{N})\, dt = 0. \tag{4.11}$$

Replace the given x by $x + \eta s$, where the scalar s is chosen so that

$$(x - [\![\mathrm{N} \cup \eta]\!]_+) \cap S = \varnothing.$$

The integral in (4.11) is unaffected by this change.

With $\sigma \equiv -1$, Theorem 3.15, part (b), implies that $M(x \mid \lambda, \mathrm{N} \cup \eta) = 0$. Thus

$$0 = \sum \lambda(s)(T_\eta * T_{\mathrm{N}})(s-x) = \int_0^\infty \sum \lambda(s)\, T_{\mathrm{N}}(s - x - \eta t)\, dt$$

$$= \int_0^\infty M(x + \eta t \mid \lambda, \mathrm{N})\, dt.$$

If $t<0$, then

$$(x+\eta t-[\![N]\!]_+)\cap S=\varnothing,$$

so that $M(x+\eta t \mid \lambda, \mathrm{N})=0$. The claim (4.11) follows.

By Fubini's theorem, the integral in Theorem 4.10 is

$$\int_{[0,\,\infty)^{\mathrm{K}\backslash\eta}} \int_{-\infty}^{\infty} M(x+(\mathrm{K}\backslash\eta)s+\eta t)\,dt\,ds,$$

and, by (4.11), this is zero. ∎

COROLLARY 4.12. *Let* N *and* $\mathrm{N}\cup\eta$ *be complete directional matrices and let* λ *be a* $(\mathrm{N}\cup\eta)$*th difference. Then*

$$\int_{\mathbb{R}^d} M(\cdot \mid \lambda, \mathrm{N})=0.$$

Proof. Complete η to a basis K of $\mathbb{R}^d$. ∎

We shall say that the Nth difference λ is **affine** if $\int M(\cdot \mid \lambda, \mathrm{N})=1$. It will be shown that the affine Nth differences are useful in approximating $D_{\mathrm{N}} f$.

The following result generalizes (1.6).

LEMMA 4.13. *Let* N *be a complete directional matrix, and let* λ *be an affine* N*th difference. If* $M(\cdot \mid \lambda, \mathrm{N})$ *is nonnegative and has connected support, then for every* $f\in C^{\mathrm{N}}(\mathscr{U})$ *there exists* $\xi\in\operatorname{supp} M$ *such that*

$$\lambda f=D_{\mathrm{N}} f(\xi).$$

Proof. Since $M\geqslant 0$ and $\int M=1$, Corollary 3.16 implies that λf lies between the maximum and minimum of $D_{\mathrm{N}} f$ over the support of M. Therefore each of the disjoint open sets

$$\{x\colon D_{\mathrm{N}} f(x)>\lambda f\} \qquad \text{and} \qquad \{x\colon D_{\mathrm{N}} f(x)<\lambda f\}$$

has nonempty intersection with supp M. Since supp M is connected, there exists some point ξ in supp M which does not lie in either of these sets, as desired. ∎

The next results show that an affine Nth difference can approximate D_{N} without the restriction that its representer be nonnegative.

LEMMA 4.14. *Let* $\mathrm{N}\in\mathbb{D}^{d\times m}$ *be complete and let* λ *be an affine* N*th difference. For* $h>0$, *define*

$$\lambda_h\colon f\mapsto h^{-m}\lambda f(\cdot h).$$

If $f \in C^{\mathrm{N}}(\bar{\mathcal{N}})$ *for some neighborhood* $\mathcal{N}$ *of the origin, and if* ω_{N} *is the modulus of continuity of* $D_{\mathrm{N}} f$ *on* $\bar{\mathcal{N}}$, *then*

$$|\lambda_h f - D_{\mathrm{N}} f(0)| \leqslant c\omega_{\mathrm{N}}(h)$$

for sufficiently small h and for some c depending only on λ.

Proof. Because $\int M(\cdot \mid \lambda, \mathrm{N}) = 1$ and $D_{\mathrm{N}} f(\cdot h) = h^m (D_{\mathrm{N}} f)(\cdot h)$, we have

$$\lambda_h f - D_{\mathrm{N}} f(0) = \int_{\mathbb{R}^d} M(t \mid \lambda, \mathrm{N}) \{(D_{\mathrm{N}}) f(th) - D_{\mathrm{N}} f(0)\}\, dt.$$

Letting $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^d$, the absolute value of the above cannot exceed

$$\max\{\omega_{\mathrm{N}}(\|th\|) \colon t \in \operatorname{supp} M\} \int_{\mathbb{R}^d} |M|.$$

The proof is completed by taking an integer n such that $\|t\| \leqslant n$ for all $t \in \operatorname{supp} M$ and by using $\omega_{\mathrm{N}}(nh) \leqslant n\omega_{\mathrm{N}}(h)$ [16]. ▮

Using shifts, one can similarly use λ to approximate $D_{\mathrm{N}} f(x)$ for x other than zero.

The next theorem shows that this convergence can be accelerated using centered affine Nth differences.

THEOREM 4.15. *Let* $\mathrm{N} \in \mathbb{D}^{d \times m}$ *be complete and let* λ *be an affine* N*th difference. Define* $\xi \in \mathbb{R}^d$ *by*

$$\xi := \int_{\mathbb{R}^d} M(t \mid \lambda, \mathrm{N})\, t\, dt;$$

for h positive, define

$$\lambda_h^{\xi} \colon f \mapsto h^{-m} \lambda f(\xi + (\cdot - \xi)\, h).$$

If $D_{\mathrm{N}} f \in C^2(\bar{\mathcal{N}})$ *for some neighborhood* $\mathcal{N}$ *of* ξ, *then*

$$|\lambda_h^{\xi} f - D_{\mathrm{N}} f(\xi)| = O(h^2).$$

Proof. Restrict h so that $\operatorname{supp} \lambda_h^{\xi} \subset \mathcal{N}$.

As in the proof of Lemma 4.14, we have

$$\lambda_h^{\xi} f - D_{\mathrm{N}} f(\xi) = \int_{\mathbb{R}^d} M(t \mid \lambda, \mathrm{N}) \{(D_{\mathrm{N}} f)(\xi + (t - \xi)\, h) - D_{\mathrm{N}} f(\xi)\}\, dt.$$

Using the smoothness of $D_{\mathrm{N}} f$, and letting "grad" denote the gradient operator, the above can be rewritten as

$$\int_{\mathbb{R}^d} M(t \mid \lambda, \mathrm{N})(\operatorname{grad} D_{\mathrm{N}} f(\xi))^{\top} (t-\xi)\, h\, dt$$

$$+ O(\max\{\, \|(t-\xi)\, h\|^2 \colon t \in \operatorname{supp} M\}\,).$$

The integral above is zero by the choice of ξ, and the second term equals $O(h^2)$, as desired. ▮

Lemma 4.14 and Theorem 4.15 has immediate extensions to the case that N is not complete.

Given a continuous function h of compact support, there might not exist a compactly supported Nth antiderivative of h, since that would require that h have integral 0 over $\mathbb{R}^d$. In the following corollary to Lemma 3.12, we construct a compactly supported function whose Nth derivative agrees with h on a cone.

LEMMA 4.16. *Let $x \in \mathbb{R}^d$ and let h be any continuous function of compact support. If* N *is directional matrix, then there exists $f \in C_c^{\mathrm{N}}(\mathbb{R}^d)$ such that $h(y) = D_{\mathrm{N}} f(y)$ for all $y \in x + [\![\mathrm{N}]\!]_+$.*

Proof. Without loss of generality, $x = 0$. By replacing N by aN for a large positive scalar a, we may also assume that

$$([\![\mathrm{N}]\!]_+ + v) \cap \operatorname{supp} h = \varnothing \tag{4.17}$$

for every v in N.

Define $f := (-1)^{\#\mathrm{N}}\, T_{\mathrm{N}} * (\nabla_{\mathrm{N}} h)$. Then, by Lemma 3.12, $D_{\mathrm{N}} f = (-1)^{\#\mathrm{N}}\, \nabla_{\mathrm{N}} h$, and, by (4.4) and (4.17), $(-1)^{\#\mathrm{N}}\, \nabla_{\mathrm{N}} h$ and h are identical on $[\![\mathrm{N}]\!]_+$.

Lemma 3.12 guarantees that $f \in C^{\mathrm{N}}(\mathbb{R}^d)$. We need only show that f has compact support.

By its definition,

$$f(t) = \langle T_{\mathrm{N}}(t - \cdot), \nabla_{\mathrm{N}} h \rangle$$

$$= \Big\langle T_{\mathrm{N}}(t - \cdot), \sum_{\mathbb{Z}_+^{\mathrm{N}}} \nabla_{\mathrm{N}}(\beta)\, h(\cdot + \mathrm{N}\beta) \Big\rangle$$

$$= \Big\langle \sum_{\mathbb{Z}_+^{\mathrm{N}}} \nabla_{\mathrm{N}}(\beta)\, T_{\mathrm{N}}(t - \cdot + \mathrm{N}\beta), h \Big\rangle.$$

By (4.6), this is

$$\langle B_{\mathrm{N}}(\cdot - t), h \rangle.$$

Since B_{N} and h are of compact support, this is zero for sufficiently large t, completing the proof. ▮

Like Theorem 3.5, the next result characterizes the Nth differences.

THEOREM 4.18. *Let* $\mathrm{N} \in \mathbb{D}^{d \times m}$ *and let* λ *be a linear combination of finitely many point evaluations with support* S. *Then* λ *is an* N*th difference if and only if* $\lambda|_{(s:v)}$ *is a* $v^{\alpha(v)}$*th difference for every* v *in* N *and* s *in* S.

Proof. Half of this theorem is proven in Theorem 3.2, part (a). It remains to prove the converse.

Assume that $\lambda|_{(s:v)}$ is a $v^{\alpha(v)}$th difference for every v and s. The proof that λ is an Nth difference is by induction on the number of distinct elements in N, the simplest case being trivial. Assume that the result is known for directional matrices having fewer distinct elements than N, and let κ and η be different members of N. Define $\mathrm{K} := \mathrm{N} \backslash \eta^{\alpha(\eta)}$ and $\mathrm{H} := \mathrm{N} \backslash \kappa^{\alpha(\kappa)}$. By the induction hypothesis, λ is both a Kth and an Hth difference. Therefore, by Corollary 3.17, there exist distributions $M_{\mathrm{K}}(t) := \lambda T_{\mathrm{K}}(\cdot - t)$ and $M_{\mathrm{H}}(t) := \lambda T_{\mathrm{H}}(\cdot - t)$, supported on $[\![S]\!]_1$, so that

$$\lambda f = \langle D_{\mathrm{K}} f, M_{\mathrm{K}} \rangle = \langle D_{\mathrm{H}} f, M_{\mathrm{H}} \rangle$$

for any f in $C_c^{\mathrm{N}}(\mathbb{R}^d)$. Integrating by parts,

$$(-1)^{\alpha(\eta)} \langle D_{\mathrm{N}} f, M_{\mathrm{K}} * T_{\eta^{\alpha(\eta)}} \rangle = (-1)^{\alpha(\kappa)} \langle D_{\mathrm{N}} f, M_{\mathrm{H}} * T_{\kappa^{\alpha(\kappa)}} \rangle.$$

The convolution on the left is supported on $[\![S]\!]_1 + [\![\eta]\!]_+$; the one on the right is supported on $[\![S]\!]_1 + [\![\kappa]\!]_+$. Thus the supports of both are contained in $x + [\![N]\!]_+$ for some $x \in \mathbb{R}^d$.

Corollary 4.16 guarantees that, by choosing f in $C_c^{\mathrm{N}}(\mathbb{R}^d)$ correctly, the restriction of $D_{\mathrm{N}} f$ to $x + [\![N]\!]_+$ can be made equal to any hat function with small support. It follows that, in the distributional sense,

$$(-1)^{\alpha(\eta)} M_{\mathrm{K}} * T_{\eta^{\alpha(\eta)}} = (-1)^{\alpha(\kappa)} M_{\mathrm{H}} * T_{\kappa^{\alpha(\kappa)}} =: M_{\mathrm{N}},$$

and that this distribution is supported on the compact set $[\![S]\!]_1 + [\![\kappa]\!]_+ \cap [\![S]\!]_1 + [\![\eta]\!]_+$.

To show that λ is an Nth difference, let $p \in \Pi_{\mathrm{N}}$, and let $f \in C_c^{\infty}(\mathbb{R}^d)$ agree with p on an open disk containing S and supp M_{N}. Then $\lambda p = \lambda f = \langle D_{\mathrm{N}} f, M_{\mathrm{N}} \rangle = 0$. By Theorem 3.5, λ is an Nth difference, completing the inductive step. ▮

COROLLARY 4.19. *Let S be a finite set in $\mathbb{R}^d$ and let* H *be the set of distinct elements from the directional matrix* N. *Then*

$$\Pi_{\mathrm{N}}(S) = \sum_{\mathrm{H}} \Pi_{\nu^{\alpha(\nu)}}(S).$$

That is, if f is a function defined on S, then there exists a polynomial in Π_{N} agreeing with f on S if and only if there exist polynomials p_ν in $\Pi_{\nu^{\alpha(\nu)}}$ such that $\sum p_\nu$ agrees with f on S.

Proof. Clearly, $\Pi_{\mathrm{N}}(S) \supset \sum \Pi_{\nu^{\alpha(\nu)}}(S)$.

Since both spaces are finite-dimensional, it will suffice to show that, if λ is a linear functional on $\mathbb{R}^S$ that vanishes on $\sum \Pi_{\nu^{\alpha(\nu)}}(S)$, then λ also vanishes on $\Pi_{\mathrm{N}}(S)$.

Since $\Pi_{\kappa^{\alpha(\kappa)}} \subset \sum \Pi_{\nu^{\alpha(\nu)}}$ for every κ in H, if $\lambda \perp \sum \Pi_{\nu^{\alpha(\nu)}}(S)$, then $\lambda \perp \Pi_{\kappa^{\alpha(\kappa)}}(S)$. By Theorem 3.5 and 4.18, $\lambda \perp \Pi_{\mathrm{N}}(S)$. ∎

Given a set S satisfying the necessary condition of Corollary 3.3, one can search for the Nth differences supported on S as follows.

By Theorems 3.5 and 4.18, λ is an Nth difference if and only if $\lambda|_{(s:\nu)} \perp \Pi_{\nu^{\alpha(\nu)}}$. For each s and ν, choose a basis P^s_ν of the univariate polynomials of degree less than $\alpha(\nu)$. Then the set

$$\{p \circ \nu^{\top}|_{(s:\nu)} : p \in P^s_\nu\}$$

is a basis for $\Pi_{\nu^{\alpha(\nu)}}(s:\nu)$. Therefore, for λ to be an Nth difference, it is necessary and sufficient that λ annihilate the set of functions

$$\bigcup_{S,\,\mathrm{N}} \{(p \circ \nu^{\top})\, \chi_{(s:\nu)} : p \in P^s_\nu\}.$$

This amounts to a homogeneous system of linear equations to be satisfied by a nontrivial λ in $\mathbb{R}^S$. One is left free to choose P^s_ν to depend on s so that this system might be more easily solved.

Note added in proof. For an introduction to a new multivariate divided difference, different from the topic of this paper, see the recent paper of de Boor [21].

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